

BIHARMONIC PNMC SUBMANIFOLDS IN SPHERES

A. BALMUŞ, S. MONTALDO, AND C. ONICIUC

ABSTRACT. We obtain several rigidity results for biharmonic submanifolds in \mathbb{S}^n with parallel normalized mean curvature vector field. We classify biharmonic submanifolds in \mathbb{S}^n with parallel normalized mean curvature vector field and with at most two distinct principal curvatures. In particular, we determine all biharmonic surfaces with parallel normalized mean curvature vector field in \mathbb{S}^n .

Then we investigate, for (not necessarily compact) proper biharmonic submanifolds in \mathbb{S}^n , their type in the sense of B-Y. Chen. We prove: (i) a proper biharmonic submanifold in \mathbb{S}^n is of 1-type or 2-type if and only if it has constant mean curvature $f = 1$ or $f \in (0, 1)$, respectively; (ii) there are no proper biharmonic 3-type submanifolds with parallel normalized mean curvature vector field in \mathbb{S}^n .

1. INTRODUCTION

Let $\varphi : M \rightarrow (N, h)$ be the inclusion of a submanifold M into a Riemannian manifold (N, h) . We say that the inclusion is *biharmonic*, or M is biharmonic, if its mean curvature vector field H satisfies the following equation

$$(1.1) \quad \tau_2(\varphi) = -m \left(\Delta H + \text{trace } R^N(d\varphi(\cdot), H)d\varphi(\cdot) \right) = 0,$$

where Δ denotes the rough Laplacian on sections of the pull-back bundle $\varphi^{-1}(TN)$ and R^N denotes the curvature operator on (N, h) . The section $\tau_2(\varphi)$ is called the *bitension field*.

When M is compact, the biharmonic condition arises from a variational problem for maps: for an arbitrary smooth map $\varphi : (M, g) \rightarrow (N, h)$ we define

$$E_2(\varphi) = \frac{1}{2} \int_M |\tau(\varphi)|^2 v_g,$$

where $\tau(\varphi) = \text{trace } \nabla d\varphi$ is the *tension field*. The functional E_2 is called the *bi-energy functional*. In the particular case when $\varphi : (M, g) \rightarrow (N, h)$ is a Riemannian immersion, the tension field has the expression $\tau(\varphi) = mH$ and equation (1.1) is equivalent to φ being a critical point of E_2 .

Obviously, any minimal submanifold ($H = 0$) is biharmonic. The non-harmonic biharmonic submanifolds are called *proper biharmonic*.

The study of proper biharmonic submanifolds is nowadays becoming a very active subject and its popularity initiated with the challenging conjecture of B-Y. Chen: *any biharmonic submanifold in an Euclidean space is minimal*.

Date: October 20, 2011.

2010 Mathematics Subject Classification. 58E20.

Key words and phrases. biharmonic submanifolds, finite type submanifolds.

The first author was supported by Grant POSDRU/89/1.5/S/49944, Romania. The second author was supported by Contributo d'Ateneo, University of Cagliari, Italy. The third author was supported by Grant PN-II-RU-TE-2011-3-0108, Romania.

Due to some nonexistence results (see [19, 25]) the Chen conjecture was generalized to: *any biharmonic submanifold in a Riemannian manifold with nonpositive sectional curvature is minimal*, but this was proved not to hold. Indeed, in [27] the authors constructed examples of proper biharmonic hypersurfaces in a 5-dimensional space of non-constant negative sectional curvature.

Yet, the conjecture is still open in its full generality for ambient spaces with constant nonpositive sectional curvature, although it was proved to be true in numerous cases when additional geometric properties for the submanifolds were assumed (see, for example, [3, 6, 9, 14, 17]).

By way of contrast, as we shall detail in Section 2, there are several families of examples of proper biharmonic submanifolds in the n -dimensional unit Euclidean sphere \mathbb{S}^n . For simplicity we shall denote these classes by B1, B2, B3 and B4. Nevertheless, a full understanding of the geometry of proper biharmonic submanifolds in \mathbb{S}^n has not been achieved. The goal of this paper is to continue the study of proper biharmonic submanifolds in \mathbb{S}^n that was initiated for the very first time in [19] and then developed in [2] – [7], [23, 25].

In [4] the proper biharmonic submanifolds with parallel mean curvature vector field (PMC) in \mathbb{S}^n were studied. In the first part of this paper we extend our study to biharmonic submanifolds with parallel normalized mean curvature vector field (PNMC). We recall that there exist PNMC surfaces which are not PMC (see [11, 20]) and, obviously, a PNMC submanifold is PMC if and only if it has constant mean curvature (CMC). We underline the fact that all known examples of proper biharmonic submanifolds in spheres are CMC, but there is no general result concerning the constancy of the mean curvature of proper biharmonic submanifolds in \mathbb{S}^n .

First, in Section 3, under some hypotheses on the mean curvature or on the squared norm of the Weingarten operator associated to the mean curvature vector field, we prove that compact, or complete, PNMC biharmonic submanifolds are PMC.

As we shall see in Section 4, PNMC pseudo-umbilical biharmonic submanifolds in \mathbb{S}^n are of class B3. We then study the PNMC biharmonic submanifolds in \mathbb{S}^n with at most two distinct principal curvatures in the direction of the mean curvature vector field, proving that they are CMC and belong to the classes B3 or B4 (Theorem 4.4).

The second part of the paper is devoted to finite type submanifolds. These submanifolds were introduced by B-Y. Chen (see, for example, [8, 10]) in the attempt of finding the best possible estimate of the total mean curvature of a compact submanifold in the Euclidean space. Although defined in a different manner, finite type submanifolds arise also, in a natural way, as solutions of a variational problem.

We prove that proper biharmonic submanifolds in spheres are of 1-type or 2-type if and only if they are CMC with mean curvature $f = 1$ or $f \in (0, 1)$, respectively (Theorem 5.8).

Moreover, we prove that there are no 3-type PNMC biharmonic submanifolds in \mathbb{S}^n (Theorem 5.10), obtaining the nonexistence of 3-type biharmonic hypersurfaces in \mathbb{S}^n (Corollary 5.11).

Finally, under some extra conditions (mass-symmetric and independent) on finite k -type submanifolds in \mathbb{S}^n we prove that biharmonicity implies that $k = 2$ (Proposition 5.12).

Conventions. Throughout this paper all manifolds, metrics, maps are assumed to be smooth, i.e. C^∞ . All manifolds are assumed to be connected. The following sign conventions are used

$$\Delta V = -\text{trace } \nabla^2 V, \quad R^N(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]},$$

$V \in C(\varphi^{-1}(TN))$ and $X, Y \in C(TN)$.

Acknowledgements. The authors would like to thank Professors B-Y. Chen and I. Dimitric for helpful discussions.

2. BIHARMONIC SUBMANIFOLDS

The key ingredient in the study of biharmonic submanifolds is the splitting of the bitension field with respect to its normal and tangent components.

Theorem 2.1. *The canonical inclusion $\varphi : M^m \rightarrow N^n$ of a submanifold M in a Riemannian manifold N is biharmonic if and only if the normal and the tangent components of $\tau_2(\varphi)$ vanish, i.e. respectively*

$$(2.1a) \quad \Delta^\perp H + \text{trace } B(\cdot, A_H \cdot) + \text{trace}(R^N(\cdot, H) \cdot)^\perp = 0,$$

and

$$(2.1b) \quad \begin{aligned} \frac{m}{2} \text{grad } f^2 + 2 \text{trace } A_{\nabla_{(\cdot)}^\perp H}(\cdot) + 2 \text{trace}(R^N(\cdot, H) \cdot)^\top &= \\ -\frac{m}{2} \text{grad } f^2 + 2 \text{trace}(\nabla A_H)(\cdot, \cdot) &= 0, \end{aligned}$$

where A denotes the Weingarten operator, B the second fundamental form, H the mean curvature vector field, $f = |H|$ the mean curvature function, ∇^\perp and Δ^\perp the connection and the Laplacian in the normal bundle of M in N .

This result was obtained in [10, 25] for submanifolds in space forms, and in [26] for general hypersurfaces. We note that the tangent part of $\tau_2(\varphi)$ vanishes if and only if the stress-energy tensor for biharmonic maps associated to φ vanishes (see [22, 18]). In the case the ambient space is a space form of sectional curvature c , (2.1a) – (2.1b) reduce to

Corollary 2.2 ([10, 25]). *The canonical inclusion $\varphi : M^m \rightarrow \mathbb{E}^n(c)$ of a submanifold M in the space form $\mathbb{E}^n(c)$ is biharmonic if and only if*

$$(2.2) \quad \begin{cases} \Delta^\perp H + \text{trace } B(\cdot, A_H \cdot) - mcH = 0, \\ 2 \text{trace } A_{\nabla_{(\cdot)}^\perp H}(\cdot) + \frac{m}{2} \text{grad } f^2 = 0. \end{cases}$$

Up to now there are not known examples of proper biharmonic submanifolds in a space form $\mathbb{E}^n(c)$ with $c \leq 0$, i.e. proper solutions of (2.2) with $c \leq 0$. This fact has suggested, as we have mentioned in the introduction, the generalized Chen conjecture.

If $c = 1$, the situation is rather different and the following are considered to be the *main examples* of proper biharmonic submanifolds in $\mathbb{S}^n = \mathbb{E}^n(1)$:

B1. The small hypersphere

$$\mathbb{S}^{n-1}(1/\sqrt{2}) = \left\{ (x, 1/\sqrt{2}) \in \mathbb{R}^{n+1} : x \in \mathbb{R}^n, |x|^2 = 1/2 \right\} \subset \mathbb{S}^n.$$

B2. The standard products of spheres

$$\mathbb{S}^{n_1}(1/\sqrt{2}) \times \mathbb{S}^{n_2}(1/\sqrt{2}) = \left\{ (x, y) \in \mathbb{R}^{n_1+1} \times \mathbb{R}^{n_2+1}, |x|^2 = |y|^2 = 1/2 \right\} \subset \mathbb{S}^n,$$

$$n_1 + n_2 = n - 1 \text{ and } n_1 \neq n_2.$$

B3. The minimal submanifolds M in a small hypersphere $\mathbb{S}^{n-1}(1/\sqrt{2}) \subset \mathbb{S}^n$.

B4. The minimal submanifolds $M_1^{m_1} \times M_2^{m_2}$ in $\mathbb{S}^{n_1}(1/\sqrt{2}) \times \mathbb{S}^{n_2}(1/\sqrt{2}) \subset \mathbb{S}^n$, with $n_1 + n_2 = n - 1$, $m_1 \neq m_2$.

Example B2 was found in [19], while example B1 was derived in [7]. The two families of examples described in B3 and B4 were constructed in [6]. Moreover, B3 is a consequence of the following property.

Theorem 2.3 ([6]). *Let M be a minimal submanifold in a small hypersphere $\mathbb{S}^{n-1}(a) \subset \mathbb{S}^n$, $a \in (0, 1)$. Then M is proper biharmonic in \mathbb{S}^n if and only if $a = 1/\sqrt{2}$.*

We note that the proper biharmonic submanifolds in \mathbb{S}^n obtained from minimal submanifolds of the proper biharmonic hypersphere $\mathbb{S}^{n-1}(1/\sqrt{2})$ have constant mean curvature $f = 1$.

More generally, we have the following bounds for the mean curvature of CMC proper biharmonic submanifolds in \mathbb{S}^n .

Theorem 2.4 ([24]). *Let M be a CMC proper biharmonic submanifold in \mathbb{S}^n . Then $f \in (0, 1]$. Moreover, if $f = 1$, then M is a minimal submanifold of a small hypersphere $\mathbb{S}^{n-1}(1/\sqrt{2}) \subset \mathbb{S}^n$.*

Notice also that proper biharmonic submanifolds in \mathbb{S}^n obtained from minimal submanifolds of $\mathbb{S}^{n-1}(1/\sqrt{2})$ have parallel mean curvature vector field (PMC) and are pseudo-umbilical, i.e. $A_H = f^2 \text{Id}$. In [28] it was proved that an umbilical biharmonic surface in any 3-dimensional Riemannian manifolds must be a CMC surface. This is a particular case of the following proposition.

Proposition 2.5. *Let $\varphi : M^m \rightarrow N$ be a submanifold of a given space N , $m \neq 4$. If M is pseudo-umbilical, then the tangent part of $\tau_2(\varphi)$ vanishes, i.e. (2.1b) is satisfied, if and only if M is CMC. In particular, if M is a biharmonic pseudo-umbilical submanifold of N , $m \neq 4$, then M is CMC.*

Proof. Since M is pseudo-umbilical, $A_H = f^2 \text{Id}$ and we find immediately

$$(2.3) \quad \text{trace}(\nabla A_H)(\cdot, \cdot) = \text{grad } f^2.$$

Then (2.1b) is equivalent to

$$(m - 4) \text{grad } f^2 = 0,$$

and we conclude the proof. \square

We recall that a pseudo-umbilical submanifold M^m , $m \neq 4$, of codimension two in \mathbb{S}^n is proper biharmonic if and only if it is minimal in $\mathbb{S}^{m+1}(1/\sqrt{2})$ (see [3]). Now, a natural question arises: *for arbitrary codimension, is a pseudo-umbilical proper biharmonic submanifold M^m in \mathbb{S}^n , $m \neq 4$, minimal in $\mathbb{S}^{n-1}(1/\sqrt{2})$?*

3. BIHARMONIC SUBMANIFOLDS WITH PARALLEL NORMALIZED MEAN CURVATURE VECTOR FIELD IN \mathbb{S}^n

A submanifold M in a Riemannian manifold is said to have *parallel normalized mean curvature vector field* (PNMC) if it has nowhere zero mean curvature and the unit vector field in the direction of the mean curvature vector field is parallel in the normal bundle, i.e.

$$(3.1) \quad \nabla^\perp(H/f) = 0,$$

where $f = |H|$ is a smooth and positive function. In the following, for a PNMC submanifold, we shall denote by $\xi = H/f$ the normalized mean curvature vector field and by A the Weingarten operator associated to ξ .

PNMC submanifolds generalize non-minimal PMC submanifolds. Moreover, for CMC submanifolds PNMC is equivalent to PMC. Note that, as stated in [11, 20], it is possible to find examples of PNMC submanifolds which are not PMC.

The characterization of PNMC biharmonic submanifolds in \mathbb{S}^n follows by Corollary 2.2.

Theorem 3.1. *Let $\varphi : M^m \rightarrow \mathbb{S}^n$ be a PNMC submanifold in the n -dimensional unit Euclidean sphere \mathbb{S}^n . Then M is biharmonic if and only if*

$$(3.2) \quad \begin{cases} \text{trace } B(\cdot, A_H \cdot) = \left(m - \frac{1}{f} \Delta f\right) H, \\ A_H(\text{grad } f^2) = -\frac{m}{2} f^2 \text{grad } f^2, \end{cases}$$

or, equivalently,

$$(3.3) \quad \begin{cases} \text{(i)} & \langle A, A_\eta \rangle = 0, \quad \forall \eta \in C(NM), \eta \perp \xi, \\ \text{(ii)} & \Delta f = (m - |A|^2)f, \\ \text{(iii)} & A(\text{grad } f^2) = -\frac{m}{2} f \text{grad } f^2, \end{cases}$$

where NM denotes the normal bundle of M in \mathbb{S}^n .

Proof. Let $p \in M$ and consider $\{E_i\}_{i=1}^m$ to be a local orthonormal frame field on M geodesic at p . Since M is PNMC, we have

$$(3.4) \quad \nabla_X^\perp H = \frac{1}{f} X(f)H, \quad \forall X \in C(TM).$$

From here, at p we have

$$\begin{aligned} \Delta^\perp H &= -\text{trace}(\nabla^\perp)^2 H = -\sum_{i=1}^m \nabla_{E_i}^\perp \left(\frac{1}{f} E_i(f)H \right) \\ &= \frac{1}{f} (\Delta f)H, \end{aligned}$$

which implies that the first equation of (2.2) becomes the first equation of (3.2).

From (3.4) we obtain

$$(3.5) \quad \text{trace } A_{\nabla_{(\cdot)}^\perp H}(\cdot) = \sum_{i=1}^m A_{\nabla_{E_i}^\perp H}(E_i) = \frac{1}{2f^2} A_H(\text{grad } f^2),$$

and the second equation of (2.2) becomes the second equation of (3.2).

Next, since $A_H = fA$, by considering the components of $\text{trace } B(\cdot, A_H \cdot)$, the one parallel to ξ and the one orthogonal to ξ , one verifies immediately that equations (3.3) and (3.2) are equivalent. \square

3.1. The compact case. Immediate consequences for compact PNMC biharmonic submanifolds follow from (3.3)(ii).

Corollary 3.2. *Let M be a compact PNMC biharmonic submanifold in \mathbb{S}^n .*

- (i) *If $|A|^2 < m$, or $|A|^2 \geq m$, on M , then M is PMC and $|A|^2 = m$.*
- (ii) *If $|A|$ is constant, then M is PMC and $|A|^2 = m$.*

From Corollary 3.2, if M is a compact PNMC biharmonic submanifold in \mathbb{S}^n , then either there exists $p \in M$ such that $|A(p)|^2 < m$, or $|A|^2 = m$.

Moreover, as a consequence of Corollary 3.2, we shall also prove that compact PNMC biharmonic submanifolds in \mathbb{S}^n , with a supplementary bounding condition on the mean curvature, are PMC. First we recall that a compact proper biharmonic submanifold in \mathbb{S}^n admits at least one point p with $f(p) \leq 1$ (see [4]), thus when considering the hypothesis $f^2 \geq \frac{4}{m}$ we have to assume $m \geq 5$ (if $m = 4$, then $f = 1$).

Proposition 3.3. *Let M^m be a compact PNMC biharmonic submanifold in \mathbb{S}^n , $m \geq 5$. If the mean curvature of M satisfies $f^2 \geq \frac{4}{m}$, then M is PMC.*

Proof. We will show that, in the given hypotheses, we have $|A|^2 \geq m$ on M , thus, by Corollary 3.2, M is PMC.

Let $p_0 \in M$ be arbitrarily fixed. We have two cases.

Case 1. If $\text{grad}_{p_0} f \neq 0$, since M is PNMC biharmonic, from (3.3)(iii) we have

$$(3.6) \quad |A(p_0)|^2 \geq \frac{m^2}{4} f^2(p_0) \geq m.$$

Case 2. Consider now the case when $\text{grad}_{p_0} f = 0$. If there exists an open subset $U \subset M$, $p_0 \in U$, such that $\text{grad } f|_U = 0$, then equation (3.3)(ii) implies that $|A|^2 = m$ on U . Otherwise, p_0 is a limit point for the set $V = \{p \in M : \text{grad}_p f \neq 0\}$. By Case 1 we have $|A(p)| \geq m$, for all $p \in V$. Therefore, we obtain $|A(p_0)|^2 \geq m$, and the proof is completed. \square

By Proposition 3.3 and Theorem 3.11 in [4] we get the following.

Corollary 3.4. *Let M^m be a compact PNMC biharmonic submanifold in \mathbb{S}^n such that $f^2 \geq \frac{4}{m}$, $m \in \{4, 5, 6, 7\}$. Then M is minimal in $\mathbb{S}^{n-1}(1/\sqrt{2})$.*

Since hypersurfaces with nowhere zero mean curvature are PNMC submanifolds, we have the following result.

Corollary 3.5. *Let M^m be a compact biharmonic hypersurface in \mathbb{S}^{m+1} , $m \geq 5$. If the mean curvature of M satisfies $f^2 \geq \frac{4}{m}$, then M is CMC.*

3.2. The non-compact case. For the non-compact case, if M^m is a PNMC biharmonic submanifold in \mathbb{S}^n such that $|A|^2 \geq m$, then f is a subharmonic function and therefore either f is constant, or f can not attain its maximum. In the following we shall prove that, under some additional hypotheses, the latter case can not occur.

We shall need the following theorem.

Theorem 3.6 (Omori-Yau Maximum Principle, [32]). *If M^m is a complete Riemannian manifold with Ricci curvature bounded from below, then for any function $u \in C^2(M)$, bounded from above, there exists a sequence of points $\{p_k\}_{k \in \mathbb{N}} \subset M$ satisfying*

$$\lim_{k \rightarrow \infty} u(p_k) = \sup_M u, \quad |\text{grad}_{p_k} u| < \frac{1}{k}, \quad \Delta u(p_k) > -\frac{1}{k}.$$

Now we can prove our result.

Proposition 3.7. *Let M^m be a complete PNMC biharmonic submanifold with non-negative Ricci curvature in \mathbb{S}^n . If $|A|$ is constant and the mean curvature of M satisfies $f^2 \geq \frac{4}{m}$, then M is PMC and $|A|^2 = m$.*

Proof. By an argument similar to that in the proof of Proposition 3.3, we get $|A|^2 \geq m$ on M , and (3.3)(ii) implies $\Delta f \leq 0$ on M .

On the other hand, we are in the hypotheses of Theorem 3.6, and thus there exists $\{p_k\}_k \subset M$ satisfying $\Delta f(p_k) > -\frac{1}{k}$. Therefore, $\lim_{k \rightarrow \infty} \Delta f(p_k) = 0$. Using this in (3.3)(ii), since $f^2 \geq \frac{4}{m}$, we obtain $|A|^2 = m$ and $\Delta f = 0$ on M .

Now, using the fact that on a complete manifold with non-negative Ricci curvature there are no non-constant bounded harmonic functions (see [32]), we conclude. \square

For hypersurfaces this result is expressed as follows.

Corollary 3.8. *Let M^m be a complete biharmonic hypersurface with non-negative Ricci curvature in \mathbb{S}^{m+1} . If $|A|$ is constant and the mean curvature of M satisfies $f^2 \geq \frac{4}{m}$, then M is CMC and $|A|^2 = m$. In this case, $m \geq 8$ and $f^2 < 1$.*

4. PNMC BIHARMONIC SUBMANIFOLDS IN \mathbb{S}^n WITH AT MOST TWO DISTINCT PRINCIPAL CURVATURES

Inspired by the case of hypersurfaces (see [3]), we intend to study PNMC biharmonic submanifolds in \mathbb{S}^n by taking into account the number of distinct principal curvatures in the direction of the mean curvature vector field.

Proposition 4.1. *Let M^m , $m \geq 2$, be a pseudo-umbilical PNMC submanifold in \mathbb{S}^n , then M is PMC. Moreover, M is minimal in $\mathbb{S}^{n-1}(a) \subset \mathbb{S}^n$, for some $a \in (0, 1)$.*

Proof. For any submanifold in \mathbb{S}^n by the Codazzi equation, we have

$$(4.1) \quad 2 \text{trace}(\nabla A_H)(\cdot, \cdot) = m \text{grad } f^2 + 2 \text{trace } A_{\nabla_{(\cdot)}^\perp H}(\cdot)$$

Now, taking into account (2.3) and (3.5), (4.1) becomes

$$(m - 1) \text{grad } f^2 = 0.$$

Thus M is PMC and, using a result of B-Y. Chen (see [10, pag. 133]), follows that M is minimal in $\mathbb{S}^{n-1}(a) \subset \mathbb{S}^n$, for some $a \in (0, 1)$. \square

Combining Proposition 4.1 and Theorem 2.3, follows that

Proposition 4.2. *Any PNMC pseudo-umbilical biharmonic submanifold in \mathbb{S}^n is minimal in $\mathbb{S}^{n-1}(1/\sqrt{2})$.*

Thus, the next step consists in classifying the PNMC biharmonic submanifolds in \mathbb{S}^n with at most two distinct principal curvatures in the direction of H . Notice that any hypersurface with nowhere zero mean curvature is PNMC, and the classification of proper biharmonic hypersurfaces with at most two distinct principal curvatures was achieved in [3]. In order to obtain the desired general classification, we first have to prove the following result.

Theorem 4.3. *Let M^m be a PNMC biharmonic submanifold in \mathbb{S}^n with at most two distinct principal curvatures in the direction of H . Then M is PMC.*

Proof. It is sufficient to prove that f , which is a positive function on M , is constant. Suppose that $f \neq \text{constant}$. Then, there exists $p \in M$ such that $\text{grad}_p f \neq 0$, thus there exists U a neighborhood of p in M such that $\text{grad} f \neq 0$ on U . Taking into account Proposition 4.2, U can not be made out only of pseudo-umbilical points. We can then assume that there exists a point $q \in U$ which is not pseudo-umbilical. Then, eventually by restricting U , we can assume that $A \neq f \text{Id}$ at every point of U , thus A has exactly two distinct principal curvatures on U .

From (3.3) (iii) we have

$$A(\text{grad} f) = -\frac{m}{2}f \text{grad} f,$$

i.e. $E_1 = \frac{\text{grad} f}{|\text{grad} f|}$ is a principal direction for A , with principal curvature

$$(4.2) \quad k_1 = -\frac{m}{2}f.$$

Recall that, as A has exactly two distinct principal curvatures, the multiplicities of its principal curvatures are constant and the principal curvatures are smooth (see [29]). Thus A is diagonalizable with respect to a local orthonormal frame field, and we can chose it to have the first field equal to E_1 , i.e. the frame field is $\{E_1, \dots, E_m\}$. We then have $A(E_i) = \bar{k}_i E_i$, $i = 1, \dots, m$, where not all the \bar{k}_i 's are different and, by construction, $\bar{k}_1 = k_1$. Since $\langle E_\alpha, E_1 \rangle = 0$, we have on U

$$(4.3) \quad E_\alpha(f) = 0, \quad \forall \alpha = 2, \dots, m.$$

We shall use the connection equations with respect to the frame field $\{E_1, \dots, E_m\}$,

$$(4.4) \quad \nabla_{E_i} E_j = \omega_j^k(E_i) E_k.$$

Let us first prove that the multiplicity of k_1 is $m_1 = 1$. Suppose that there exists $\alpha \in \{2, \dots, m\}$, such that $\bar{k}_\alpha = k_1$ on U . Since $\nabla^\perp \xi = 0$, the Codazzi equation for A is

$$(4.5) \quad (\nabla_{E_i} A)(E_j) = (\nabla_{E_j} A)(E_i), \quad \forall i, j = 1, \dots, m.$$

By using (4.4), the Codazzi equation becomes

$$(4.6) \quad E_i(\bar{k}_j)E_j + \sum_{\ell=1}^m (\bar{k}_j - \bar{k}_\ell)\omega_j^\ell(E_i)E_\ell = E_j(\bar{k}_i)E_i + \sum_{\ell=1}^m (\bar{k}_i - \bar{k}_\ell)\omega_i^\ell(E_j)E_\ell.$$

Putting $i = 1$ and $j = \alpha$ in (4.6) and taking the scalar product with E_α we obtain $E_1(k_1) = 0$, which, together with (4.2) and (4.3), gives $f = \text{constant}$, thus we have a contradiction.

Thus $\bar{k}_1 = k_1$ and $\bar{k}_\alpha = k_2$, for all $\alpha = 2, \dots, m$, and since $\text{trace} A = mf$, we get

$$(4.7) \quad k_2 = \frac{3}{2} \frac{m}{m-1} f.$$

Putting $i = 1$ and $j = \alpha$ in (4.6) and taking the scalar product with E_α , E_β , $\beta \neq \alpha$, and E_1 , respectively, one gets

$$(4.8a) \quad \omega_1^\alpha(E_\alpha) = -\frac{3}{m+2} \frac{E_1(f)}{f},$$

$$(4.8b) \quad \omega_1^\alpha(E_\beta) = 0,$$

$$(4.8c) \quad \omega_1^\alpha(E_1) = 0,$$

for all $\alpha, \beta = 2, \dots, m$, $\alpha \neq \beta$.

Consider $\{\eta_{m+1} = \xi, \eta_{m+2}, \dots, \eta_n\}$ to be an orthonormal normal frame field on U in \mathbb{S}^n and denote by $A_a = A_{\eta_a}$, $a = m+2, \dots, n$. Since $\nabla^\perp \xi = 0$, from the Ricci equation of U in \mathbb{S}^n , we have

$$A \circ A_a = A_a \circ A, \quad \forall a = m+2, \dots, n.$$

Since k_1 has multiplicity 1, it follows directly that E_1 is a principal direction for A_a , for all $a = m+2, \dots, n$. Fix $a \in \{m+2, \dots, n\}$ and denote $A_a(E_1) = \lambda_a E_1$ on U . From (3.3) (i), we have that $\sum_{i=1}^m \langle A(E_i), A_a(E_i) \rangle = 0$ and this leads to

$$(k_1 - k_2)\lambda_a + k_2 \text{trace } A_a = 0.$$

Since $\text{trace } A_a = m \langle H, \eta_a \rangle = 0$, we conclude that $\lambda_a = 0$, i.e.

$$(4.9) \quad A_a(E_1) = 0, \quad \forall a = m+2, \dots, n.$$

We now express the Gauss equation for U in \mathbb{S}^n ,

$$(4.10) \quad \begin{aligned} \langle R^{\mathbb{S}^n}(X, Y)Z, W \rangle &= \langle R(X, Y)Z, W \rangle \\ &+ \langle B(X, Z), B(Y, W) \rangle - \langle B(X, W), B(Y, Z) \rangle, \end{aligned}$$

with $X = W = E_1$ and $Y = Z = E_\alpha$. Using (4.9) one obtains

$$B(E_1, E_\alpha) = 0, \quad B(E_1, E_1) = k_1 \xi, \quad \langle B(E_\alpha, E_\alpha), B(E_1, E_1) \rangle = k_1 k_2.$$

From (4.4), (4.8b), (4.8c), and using $\omega_j^k = -\omega_k^j$, the curvature term is

$$\langle R(E_1, E_\alpha)E_\alpha, E_1 \rangle = -E_1(\omega_1^\alpha(E_\alpha)) - (\omega_1^\alpha(E_\alpha))^2.$$

Finally, (4.10) and (4.8a) imply

$$(4.11) \quad f E_1(E_1(f)) = \frac{m+2}{3} f^2 - \frac{m^2(m+2)}{4(m-1)} f^4 + \frac{m+5}{m+2} (E_1(f))^2.$$

From (4.2) and (4.7), we have

$$(4.12) \quad |A|^2 = k_1^2 + (m-1)k_2^2 = \frac{m^2(m+8)}{4(m-1)} f^2.$$

Moreover, using (4.3), (4.4) and (4.8a) the Laplacian of f becomes

$$(4.13) \quad \begin{aligned} \Delta f &= -E_1(E_1(f)) - \sum_{\alpha=2}^m E_\alpha(E_\alpha(f)) + (\nabla_{E_1} E_1)f + \sum_{\alpha=2}^m (\nabla_{E_\alpha} E_\alpha)f \\ &= -E_1(E_1(f)) + \sum_{\alpha=2}^m \omega_\alpha^1(E_\alpha) E_1(f) \\ &= -E_1(E_1(f)) + \frac{3(m-1)}{m+2} \frac{(E_1(f))^2}{f}. \end{aligned}$$

From (3.3) (ii), by substituting (4.12) and (4.13), we get

$$(4.14) \quad f E_1(E_1(f)) = -m f^2 + \frac{m^2(m+8)}{4(m-1)} f^4 + \frac{3(m-1)}{m+2} (E_1(f))^2.$$

Consider now $\gamma = \gamma(u)$ to be an arbitrary integral curve of E_1 in U . Along γ we have $f = f(u)$ and we set $w = (E_1(f))^2 = (f')^2$. Then $dw/df = 2f''$, and (4.11) and (4.14) become

$$(4.15) \quad \begin{cases} \frac{1}{2} f \frac{dw}{df} = \frac{m+2}{3} f^2 - \frac{m^2(m+2)}{4(m-1)} f^4 + \frac{m+5}{m+2} w, \\ \frac{1}{2} f \frac{dw}{df} = -m f^2 + \frac{m^2(m+8)}{4(m-1)} f^4 + \frac{3(m-1)}{m+2} w. \end{cases}$$

By subtracting the two equations we find two cases.

If $m = 4$, then

$$-\frac{2(2m+1)}{3f^2} f^2 + \frac{m^2(m+5)}{2(m-1)} f^4 = 0,$$

thus f is constant.

If $m \neq 4$, then

$$w = \frac{(m+2)(2m+1)}{3(m-4)} f^2 - \frac{m^2(m+2)(m+5)}{4(m-4)(m-1)} f^4.$$

Differentiating with respect to f and replacing this in the second equation of (4.15), we get

$$\frac{(m-1)(m+5)}{3} f^2 + \frac{3m^2(2m+1)}{4(m-1)} f^4 = 0.$$

Therefore f is constant along γ , thus $\text{grad } f = 0$ along γ and we have a contradiction. \square

As a consequence of Theorem 4.3 we have the following rigidity result.

Theorem 4.4. *Let M^m be a PNMC biharmonic submanifold in \mathbb{S}^n with at most two distinct principal curvatures in the direction of H . Then either M is minimal in $\mathbb{S}^{n-1}(1/\sqrt{2})$, or locally,*

$$M = M_1^{m_1} \times M_2^{m_2} \subset \mathbb{S}^{n_1}(1/\sqrt{2}) \times \mathbb{S}^{n_2}(1/\sqrt{2}) \subset \mathbb{S}^n,$$

where M_i is a minimal submanifold of $\mathbb{S}^{n_i}(1/\sqrt{2})$, $i = 1, 2$, $m_1 + m_2 = m$, $m_1 \neq m_2$, $n_1 + n_2 = n - 1$.

Proof. From Theorem 4.3 we conclude that M is PMC. Moreover, since A_H has at most two distinct principal curvatures in the direction of H , from Proposition 3.19 in [4], we get that $\nabla A_H = 0$ and the conclusion follows by applying Theorem 3.16 in [4]. \square

Moreover, as a corollary of Theorem 4.4, the following rigidity result, which generalizes Theorem 5.6 in [3], is valid.

Corollary 4.5. *Let M^2 be a PNMC biharmonic surface in \mathbb{S}^n . Then M is minimal in $\mathbb{S}^{n-1}(1/\sqrt{2})$.*

Remark 4.6. (i) In [11] it was proved that, in general, a PNMC analytic surface in \mathbb{S}^n is either minimal in a small hypersphere of \mathbb{S}^n , and therefore it is PMC, or it lies in a 4-dimensional great sphere $\mathbb{S}^4 \subset \mathbb{S}^n$. Notice that with no analyticity condition, by Corollary 4.5, the supplementary hypothesis that the surface is biharmonic leads only to the first case.

(ii) For the particular case of PNMC biharmonic surfaces in \mathbb{S}^4 we can give a different proof for Theorem 4.4. Indeed, using the codimension reduction result of Erbacher ([16]), one can prove that the surface lies in a great hypersphere \mathbb{S}^3 of \mathbb{S}^4 and, therefore it must have constant mean curvature.

(iii) We can slightly relax the hypotheses of Theorem 4.4, obtaining the same result, in the following way. By the unique continuation property for biharmonic maps (see [24]), if M is a proper biharmonic submanifold in \mathbb{S}^n , then H is nowhere zero on an open dense subset $W \subset M$. If we assume that $\nabla^\perp(H/f) = 0$ on W and A_H has at most two distinct principal curvatures everywhere on W , then by Theorem 4.4 we get $\nabla^\perp H = 0$ on W . By continuity we obtain $\nabla^\perp H = 0$ on M .

5. ON THE TYPE OF BIHARMONIC SUBMANIFOLDS IN \mathbb{S}^n

Definition 5.1 ([8, 10]). A submanifold $M \subset \mathbb{R}^{n+1}$ is called of *finite type* if its inclusion $\phi : M \rightarrow \mathbb{R}^{n+1}$ can be expressed as a finite sum of \mathbb{R}^{n+1} -valued eigenmaps of the Laplacian Δ of M , i.e.

$$(5.1) \quad \phi = \phi_0 + \phi_{t_1} + \dots + \phi_{t_k},$$

where $\phi_0 \in \mathbb{R}^{n+1}$ is a constant vector, $\phi_{t_i} : M \rightarrow \mathbb{R}^{n+1}$ are non-constant maps satisfying $\Delta \phi_{t_i} = \lambda_{t_i} \phi_{t_i}$, $i = 1, \dots, k$. If, in particular, all eigenvalues λ_{t_i} are assumed to be mutually distinct the submanifold is said to be of *k-type* and (5.1) is called the *spectral decomposition* of ϕ .

Remark 5.2. If M is compact the inclusion $\phi : M \rightarrow \mathbb{R}^{n+1}$ admits a unique spectral decomposition $\phi = \phi_0 + \sum_{i=1}^{\infty} \phi_i$, where ϕ_0 is the *center of mass*. Then, it is of *k-type* if only k terms of $\{\phi_i\}_{i=1}^{\infty}$ are not vanishing. In the non compact case the spectral decomposition $\phi = \phi_0 + \sum_{i=1}^{\infty} \phi_i$ is not guaranteed. Nonetheless, if Definition 5.1 is satisfied the spectral decomposition is unique. Notice also that, in the non-compact case, the harmonic component of the spectral decomposition is not necessarily constant. Finite type submanifolds with non-constant harmonic component are called *null finite type* submanifolds.

The inclusion of a *k-type* submanifold is said to be *linearly independent* if the linear subspaces

$$E_{t_i} = \text{span}\{\phi_{t_i}(u) : u \in M\}, \quad i = 1, \dots, k,$$

are linearly independent, i.e. the dimension of the subspace spanned by vectors in $\bigcup_{i=1}^k E_{t_i}$ is equal to $\sum_{i=1}^k \dim E_{t_i}$.

The following result provides us a necessary and a sufficient condition for a submanifold to be of finite type.

Theorem 5.3. [10, 13] *Let $\phi : M \rightarrow \mathbb{R}^{n+1}$ be an isometric immersion.*

- (i) *If M is of finite k-type, there exist a constant vector $\phi_0 \in \mathbb{R}^{n+1}$ and a monic polynomial with simple roots P of degree k with $P(\Delta)(\phi - \phi_0) = 0$.*

- (ii) If there exist a constant vector $\phi_0 \in \mathbb{R}^{n+1}$ and a polynomial P with simple roots such that $P(\Delta)(\phi - \phi_0) = 0$, then M is of finite k -type with $k \leq \text{degree}(P)$.

We shall also use the following version.

Theorem 5.4. [10, 13] Let $\phi : M \rightarrow \mathbb{R}^{n+1}$ be an isometric immersion.

- (i) If M is of finite k -type, there exists a monic polynomial P of degree $k - 1$ or k with $P(\Delta)H^0 = 0$.
- (ii) If there exists a polynomial P with simple roots such that $P(\Delta)H^0 = 0$, then M is of infinite type or of finite k -type with $k - 1 \leq \text{degree}(P)$.

Here H^0 denotes the mean curvature vector field of M in \mathbb{R}^{n+1} .

A well known result of T. Takahashi can be rewritten as the classification of 1-type submanifolds in \mathbb{R}^{n+1} .

Theorem 5.5 ([31]). A submanifold $M \subset \mathbb{R}^{n+1}$ is of 1-type if and only if either M is a minimal submanifold of \mathbb{R}^{n+1} , or M is a minimal submanifold of a hypersphere of \mathbb{R}^{n+1} .

Definition 5.6. A submanifold $M \subset \mathbb{S}^n$ is said to be of *finite type* if it is of finite type as a submanifold of \mathbb{R}^{n+1} , where \mathbb{S}^n is canonically embedded in \mathbb{R}^{n+1} . Moreover, a non-null finite type submanifold in \mathbb{S}^n is said to be *mass-symmetric* if the constant vector ϕ_0 of its spectral decomposition is the center of the hypersphere \mathbb{S}^n , i.e. $\phi_0 = 0$.

Remark 5.7. By Theorem 5.5, biharmonic submanifolds of class B3 are 1-type submanifolds. Indeed, the inclusion $\phi : M \rightarrow \mathbb{R}^{n+1}$ of M in \mathbb{R}^{n+1} has the spectral decomposition

$$\phi = \phi_0 + \phi_p,$$

where $\phi_0 = (0, 1/\sqrt{2})$, $\phi_p : M \rightarrow \mathbb{R}^{n+1}$, $\phi_p(x, 1/\sqrt{2}) = (x, 0)$ and $\Delta\phi_p = 2m\phi_p$.

Moreover, biharmonic submanifolds of class B4 are mass-symmetric 2-type submanifolds. Indeed, $\phi : M_1 \times M_2 \rightarrow \mathbb{R}^{n+1}$ has the spectral decomposition

$$\phi = \phi_p + \phi_q,$$

where $\phi_p(x, y) = (x, 0)$, $\phi_q(x, y) = (0, y)$, $\Delta\phi_p = 2m_1\phi_p$, $\Delta\phi_q = 2m_2\phi_q$.

Let $\varphi : M \rightarrow \mathbb{S}^n$ be a submanifold in \mathbb{S}^n and denote by $\phi = \mathbf{i} \circ \varphi : M \rightarrow \mathbb{R}^{n+1}$ the inclusion of M in \mathbb{R}^{n+1} . Denote by H the mean curvature vector field of M in \mathbb{S}^n and by H^0 the mean curvature vector field of M in \mathbb{R}^{n+1} .

The mean curvature vector fields H^0 and H are related by $H^0 = H - \phi$. Moreover, we have

$$(5.2) \quad \langle H, \phi \rangle = 0, \quad \langle H^0, H \rangle = f^2, \quad \langle H^0, \phi \rangle = -1.$$

Following [6] the bitension field of φ can be written as

$$\tau_2(\varphi) = -m\Delta H^0 + 2m^2H^0 + m^2\{2 - |H^0|^2\}\phi.$$

Thus, $\tau_2(\varphi) = 0$ if and only if

$$(5.3a) \quad \Delta H^0 - 2mH^0 + m(f^2 - 1)\phi = 0.$$

or equivalently, since $\Delta\phi = -mH^0$,

$$(5.3b) \quad \Delta^2\phi - 2m\Delta\phi - m^2(f^2 - 1)\phi = 0,$$

In [3, Theorem 3.1] we proved that CMC compact proper biharmonic submanifolds in \mathbb{S}^n are of 1-type or 2-type. This result can be generalized to the following.

Theorem 5.8. *Let $\varphi : M \rightarrow \mathbb{S}^n$ be a proper biharmonic submanifold, not necessarily compact, in the unit Euclidean sphere \mathbb{S}^n . Denote by $\phi = \mathbf{i} \circ \varphi : M \rightarrow \mathbb{R}^{n+1}$ the inclusion of M in \mathbb{R}^{n+1} , where $\mathbf{i} : \mathbb{S}^n \rightarrow \mathbb{R}^{n+1}$ is the canonical inclusion map. Then*

- (i) *M is a 1-type submanifold of \mathbb{R}^{n+1} if and only if $f = 1$. In this case, $\phi = \phi_0 + \phi_p$, $\Delta\phi_p = 2m\phi_p$, and $\phi_0 \in \mathbb{R}^{n+1}$, $|\phi_0| = 1/\sqrt{2}$.*
- (ii) *M is a 2-type submanifold if and only if $f = \text{constant}$, $f \in (0, 1)$. In this case, $\phi = \phi_p + \phi_q$, $\Delta\phi_p = m(1 - f)x_p$, $\Delta\phi_q = m(1 + f)x_q$.*

Proof. In order to prove (i), notice that the converse is obvious, by Theorem 5.5 and Theorem 2.4.

Let us suppose that M is a 1-type submanifold. From Theorem 5.4(i), it follows that there exists $a \in \mathbb{R}$ such that

$$(5.4) \quad \Delta H^0 = aH^0.$$

Equations (5.3a) and (5.4) imply

$$(2m - a)H^0 - m(f^2 - 1)\phi = 0,$$

and by considering the scalar product with H and using (5.2), since M is proper biharmonic, we get $a = 2m$ and

$$m(f^2 - 1)\phi = 0.$$

Thus $f = 1$. Now, as the map ϕ can not be harmonic, (5.3b) leads to the spectral decomposition $\phi = \phi_0 + \phi_p$, $\Delta\phi_p = 2m\phi_p$. Since $\Delta\phi = -mH^0$, taking into account the relation between H and H^0 , we obtain $2\phi_0 = \phi + H$. Since $|\phi| = 1 = f$, and H is orthogonal to ϕ , we conclude that $|\phi_0| = 1/\sqrt{2}$.

Let us now prove (ii). The converse of (ii) follows immediately. Indeed, from (5.3b), if $f = \text{constant}$, $f \in (0, 1)$, then choosing the constant vector $\phi_0 = 0$ and the polynomial with simple roots

$$P(\Delta) = \Delta^2 - 2m\Delta - m^2(f^2 - 1)\Delta^0,$$

we are in the hypotheses of Theorem 5.3(ii). Thus M is of finite k -type, with $k \leq 2$. Taking into account (i), since $f \in (0, 1)$, this implies that M is a 2-type submanifold with

$$\phi = \phi_p + \phi_q,$$

with corresponding eigenvalues $\lambda_p = m(1 - f)$, $\lambda_q = m(1 + f)$. Also, notice that

$$\phi_p = \frac{\lambda_q}{\lambda_q - \lambda_p}\phi - \frac{1}{\lambda_q - \lambda_q}\Delta\phi, \quad \phi_q = -\frac{\lambda_p}{\lambda_q - \lambda_p}\phi + \frac{1}{\lambda_q - \lambda_q}\Delta\phi,$$

which are smooth non-zero maps.

Suppose now that M is a 2-type submanifold. From Theorem 5.3(i), it follows that there exist a constant vector $\phi_0 \in \mathbb{R}^{n+1}$ and $a, b \in \mathbb{R}$ such that

$$(5.5) \quad \Delta H^0 = aH^0 + b(\phi - \phi_0).$$

Equations (5.3a) and (5.5) lead to

$$(5.6) \quad (2m - a)H^0 - (m(f^2 - 1) + b)\phi + b\phi_0 = 0.$$

We have to consider two cases.

Case 1. If $b = 0$, i.e. M is a null 2-type submanifold, by taking the scalar product

with H in (5.6) and using (5.2), since M is proper biharmonic, we get $a = 2m$ and $f = 1$. By (i), this leads to a contradiction.

Case 2. If $b \neq 0$, by taking the scalar product with $X \in C(TM)$ in (5.6), we obtain $\langle \phi_0, X \rangle = 0$, for all $X \in C(TM)$, i.e. the component of ϕ_0 tangent to M vanishes

$$(5.7) \quad (\phi_0)^\top = 0.$$

Take now the scalar product with ϕ in (5.6) and use (5.2). We obtain

$$-2m + a - m(f^2 - 1) - b + b\langle \phi_0, \phi \rangle = 0,$$

and, by differentiating,

$$(5.8) \quad m \operatorname{grad} f^2 = b \operatorname{grad} \langle \phi_0, \phi \rangle.$$

Now, by considering $\{E_i\}_{i=1}^m$ to be a local orthonormal frame field on M , we have

$$(5.9) \quad \begin{aligned} \operatorname{grad} \langle \phi_0, \phi \rangle &= \sum_{i=1}^m E_i(\langle \phi_0, \phi \rangle) E_i = \sum_{i=1}^m \langle \phi_0, \nabla_{E_i}^0 \phi \rangle E_i = \sum_{i=1}^m \langle \phi_0, E_i \rangle E_i \\ &= (\phi_0)^\top. \end{aligned}$$

This, together with equations (5.7) and (5.8), leads to $f = \text{constant}$ and using Theorem 2.4 we conclude the proof. \square

Remark 5.9. The direct implication of (i) in Theorem 5.8 can be also proved in a more geometric manner (see [1]).

We are now interested in proper biharmonic submanifolds of 3-type in spheres. In [12] it was proved that there are no CMC 3-type hypersurfaces in a hypersphere of the Euclidean space. Since the known examples of proper biharmonic hypersurfaces in spheres are CMC, one may think that there are no such hypersurfaces of type 3. Indeed, we have a more general result.

Theorem 5.10. *There exist no PNMC biharmonic 3-type submanifolds M^m in the unit Euclidean sphere \mathbb{S}^n .*

Proof. Suppose now that M is a PNMC biharmonic 3-type submanifold. From Theorem 5.3, it follows that there exist a constant vector $\phi_0 \in \mathbb{R}^{n+1}$ and $a, b, c \in \mathbb{R}$ such that

$$(5.10) \quad \Delta^2 H^0 = a \Delta H^0 + b H^0 + c(\phi - \phi_0).$$

Equations (5.3a) and (5.10) lead to

$$(5.11) \quad \Delta^2 H^0 = (2ma + b)H^0 + (c - ma(f^2 - 1))\phi - c\phi_0.$$

Now, by applying Δ to equation (5.3a) we get

$$(5.12) \quad \begin{aligned} \Delta^2 H^0 &= m^2(3 + f^2)H^0 - (m\Delta f^2 + 2m^2(f^2 - 1))\phi \\ &\quad + 2m d\phi(\operatorname{grad} f^2). \end{aligned}$$

By taking the scalar product with $\xi = H/f$ in (5.11) and (5.12) and by using (5.2), we obtain

$$(5.13) \quad -c\langle \phi_0, \xi \rangle = m^2 f^3 + (3m^2 - 2ma - b)f.$$

Consider now the scalar product with $X \in C(TM)$ in (5.11) and (5.12). This implies

$$-c\langle \phi_0, X \rangle = 2mX(f^2),$$

and, further, the component of $c\phi_0$ tangent to M is given by

$$(5.14) \quad -c(\phi_0)^\top = 2m \operatorname{grad} f^2.$$

Moreover, by taking the scalar product with an arbitrary vector field η normal to M in \mathbb{S}^n , $\eta \perp \xi$, in (5.11) and (5.12) we find

$$(5.15) \quad -c\langle \phi_0, \eta \rangle = 0.$$

Equations (5.14) and (5.15) lead to

$$(5.16) \quad -c\phi_0 = 2m \operatorname{grad} f^2 - c\langle \phi_0, \xi \rangle \xi - c\langle \phi_0, \phi \rangle \phi.$$

Differentiating (5.13), one gets

$$(5.17) \quad -c \operatorname{grad} \langle \phi_0, \xi \rangle = (3m^2 f^2 + 3m^2 - 2ma - b) \operatorname{grad} f.$$

By considering $\{E_i\}_{i=1}^m$ to be a local orthonormal frame field on M and using $\nabla^\perp \xi = 0$, (5.16), (5.13) and (5.14), we have the following

$$\begin{aligned} -c \operatorname{grad} \langle \phi_0, \xi \rangle &= -c \sum_{i=1}^m E_i(\langle \phi_0, \xi \rangle) E_i = -c \sum_{i=1}^m (\langle \nabla_{E_i}^0 \phi_0, \xi \rangle + \langle \phi_0, \nabla_{E_i}^0 \xi \rangle) E_i \\ &= -c \sum_{i=1}^m \langle \phi_0, \nabla_{E_i} \xi \rangle E_i = -c \sum_{i=1}^m \langle \phi_0, \nabla_{E_i}^\perp \xi - A(E_i) \rangle E_i \\ &= c \sum_{i=1}^m \langle (\phi_0)^\top, A(E_i) \rangle E_i = \sum_{i=1}^m \langle A(c(\phi_0)^\top), E_i \rangle E_i \\ (5.18) \quad &= -2mA(\operatorname{grad} f^2). \end{aligned}$$

Equations (5.17) and (5.18) imply

$$(5.19) \quad 2mA(\operatorname{grad} f^2) = (-3m^2 f^2 - 3m^2 + 2ma + b) \operatorname{grad} f.$$

Since M is a PNMC biharmonic submanifold, (3.3) (iii), together with (5.19), leads to

$$(m^2 f^2 + 3m^2 - 2ma - b) \operatorname{grad} f = 0,$$

on M . This implies that $\operatorname{grad} f = 0$, i.e. M is CMC. From Theorem 5.8, we have that M is a 1-type or 2-type submanifold and we get to a contradiction. \square

Since any hypersurface with nowhere zero mean curvature is PNMC we have the following.

Corollary 5.11. *There exist no biharmonic 3-type hypersurfaces M^m in the unit Euclidean sphere \mathbb{S}^{m+1} .*

Proof. Suppose that M is of 3-type. Then M is not minimal in \mathbb{S}^{m+1} , thus f is nowhere zero on an open dense subset $W \subset M$. Every connected component of W is PNMC and, by Theorem 5.10, it can not be of 3-type. This leads to a contradiction. \square

We note that the classes B3 and B4 of proper biharmonic submanifolds in spheres are linearly independent (even more, orthogonal) 2-type submanifolds. Thus it is natural to ask whether there exist proper biharmonic independent higher finite type submanifolds. We can prove the following result.

Proposition 5.12. *Let M be a proper biharmonic submanifold in \mathbb{S}^n . If M is of finite k -type, mass-symmetric and linearly independent, then $k = 2$.*

Proof. Let M be a k -type mass-symmetric submanifold in \mathbb{S}^n . Then, we have

$$\phi = \phi_{t_1} + \phi_{t_2} + \dots + \phi_{t_k},$$

where ϕ_{t_i} are non-harmonic maps satisfying $\Delta\phi_{t_i} = \lambda_{t_i}\phi_{t_i}$, and λ_{t_i} are mutually distinct, $i = 1, \dots, k$. This implies that

$$(5.20) \quad \Delta\phi = \sum_{i=1}^k \lambda_{t_i} \phi_{t_i} \quad \text{and} \quad \Delta^2\phi = \sum_{i=1}^k \lambda_{t_i}^2 \phi_{t_i}.$$

Since M is proper biharmonic, replacing (5.20) in (5.3b), we obtain

$$\sum_{i=1}^k (\lambda_{t_i}^2 - 2m\lambda_{t_i} - m^2(f^2 - 1))\phi_{t_i} = 0.$$

Using that M is independent, we get $(\lambda_{t_i}^2 - 2m\lambda_{t_i} - m^2(f^2 - 1))\phi_{t_i} = 0$ on M , for all $i = 1, \dots, k$. Since ϕ_{t_i} is non-zero on an open dense set in M , we have $\lambda_{t_i}^2 - 2m\lambda_{t_i} - m^2(f^2 - 1) = 0$ on M , for all $i = 1, \dots, k$. This implies that $f = \text{constant}$. Since ϕ is mass-symmetric, by Theorem 5.8, we conclude that $k = 2$. \square

REFERENCES

- [1] A. Balmuş, D. Fetcu, C. Oniciuc. Stability properties for biharmonic maps. Geometry Exploratory Workshop on “Differential Geometry and its Applications” Iaşi, September 2–4, 2009, 1–19, Cluj University Press, 2011.
- [2] A. Balmuş, S. Montaldo, C. Oniciuc. Biharmonic hypersurfaces in 4-dimensional space forms. Math. Nachr. 283 (2010), 1696–1705.
- [3] A. Balmuş, S. Montaldo, C. Oniciuc. Classification results for biharmonic submanifolds in spheres. Israel J. Math. 168 (2008), 201–220.
- [4] A. Balmuş, C. Oniciuc. Biharmonic submanifolds with parallel mean curvature vector field in spheres. J. Math. Anal. Appl. 386 (2012), 619–630.
- [5] A. Balmuş, C. Oniciuc. Biharmonic surfaces of \mathbb{S}^4 . Kyushu J. Math. 63 (2009), 339–345.
- [6] R. Caddeo, S. Montaldo, C. Oniciuc. Biharmonic submanifolds in spheres. Israel J. Math. 130 (2002), 109–123.
- [7] R. Caddeo, S. Montaldo, C. Oniciuc. Biharmonic submanifolds of \mathbb{S}^3 . Internat. J. Math. 12 (2001), 867–876.
- [8] B-Y. Chen. A report on submanifolds of finite type. Soochow J. Math. 22 (1996), 117–337.
- [9] B-Y. Chen. Some open problems and conjectures on submanifolds of finite type. Soochow J. Math. 17 (1991), 169–188.
- [10] B-Y. Chen. Total Mean Curvature and Submanifolds of Finite Type. Series in Pure Mathematics, 1. World Scientific Publishing Co., Singapore, 1984.
- [11] B-Y. Chen. Surfaces with parallel normalized mean curvature vector. Mh. Math. 90 (1980), 185–194.
- [12] B-Y. Chen, S.J. Li. 3-type hypersurfaces in a hypersphere. Bull. Soc. Math. Belg. Sér. B 43 (1991), no. 2, 135–141.
- [13] B-Y. Chen, M. Petrovic. On spectral decomposition of immersions of finite type. Bull. Austral. Math. Soc. 44 (1991), 117–129.
- [14] I. Dimitric. Submanifolds of \mathbb{E}^m with harmonic mean curvature vector. Bull. Inst. Math. Acad. Sinica 20 (1992), 53–65.
- [15] J. Eells, J.H. Sampson. Harmonic mappings of Riemannian manifolds. Amer. J. Math. 86 (1964), 109–160.
- [16] J. Erbacher. Reduction of the codimension of an isometric immersion. J. Differ. Geom. 5 (1971), 333–340.

- [17] T. Hasanis, T. Vlachos. Hypersurfaces in E^4 with harmonic mean curvature vector field. Math. Nachr. 172 (1995), 145–169.
- [18] G.Y. Jiang. The conservation law for 22-harmonic maps between Riemannian manifolds. Acta Math. Sinica 30 (1987), 220–225.
- [19] G.Y. Jiang. 2-harmonic isometric immersions between Riemannian manifolds. Chinese Ann. Math. Ser. A 7 (1986), 130–144.
- [20] D.S.P. Leung. The Cauchy problem for surfaces with parallel normalized mean curvature vector in a four dimensional Riemannian manifold. preprint.
- [21] E. Loubeau, S. Montaldo. Biminimal immersions. Proc. Edinb. Math. Soc. 51 (2008), 421–437.
- [22] E. Loubeau, S. Montaldo, C. Oniciuc. The stress-energy tensor for biharmonic maps. Math. Z. 259 (2008), 503–524.
- [23] N. Nakauchi, H. Urakawa. Biharmonic hypersurfaces in a Riemannian manifold with non-positive Ricci curvature. Ann. Glob. Anal. Geom. 40 (2011), 125–131.
- [24] C. Oniciuc. Tangency and Harmonicity Properties. PhD Thesis, Geometry Balkan Press 2003, <http://www.mathem.pub.ro/dgds/mono/dgdsmono.htm>
- [25] C. Oniciuc. Biharmonic maps between Riemannian manifolds. An. Stiint. Univ. Al.I. Cuza Iasi Mat (N.S.) 48 (2002), 237–248.
- [26] Y. -L. Ou. Biharmonic hypersurfaces in Riemannian manifolds. Pacific J. Math. 248 (2010), 217–232.
- [27] Y.-L. Ou, L. Tang. The generalized Chen’s conjecture on biharmonic submanifolds is false. [arXiv:1006.1838](https://arxiv.org/abs/1006.1838).
- [28] Y.-L. Ou, Z-P. Wang. Constant mean curvature and totally umbilical biharmonic surfaces in 3-dimensional geometries. J. Geom. Phys. (10) 61, (2011), 1845–1853.
- [29] P.J. Ryan. Homogeneity and some curvature conditions for hypersurfaces. Tôhoku Math. J. (2) 21 (1969), 363–388.
- [30] J. Simons. Minimal varieties in Riemannian manifolds. Ann. of Math. 88 (1968), 62–105.
- [31] T. Takahashi. Minimal immersions of Riemannian manifolds. J. Math. Soc. Japan 18 (1966), 380–385.
- [32] S.-T. Yau. Harmonic functions on complete Riemannian manifolds. Commun. Pure Appl. Math. 28 (1975), 201–228.

FACULTY OF MATHEMATICS, “AL.I. CUZA” UNIVERSITY OF IASI,
 BD. CAROL I NR. 11, 700506 IASI, ROMANIA
E-mail address: adina.balmus@uaic.ro

UNIVERSITÀ DEGLI STUDI DI CAGLIARI, DIPARTIMENTO DI MATEMATICA,
 VIA OSPEDALE 72, 09124 CAGLIARI, ITALIA
E-mail address: montaldo@unica.it

FACULTY OF MATHEMATICS, “AL.I. CUZA” UNIVERSITY OF IASI,
 BD. CAROL I NR. 11, 700506 IASI, ROMANIA
E-mail address: oniciucc@uaic.ro